On the Obfuscation Complexity of Planar Graphs

Oleg Verbitsky*

IAPMM, Lviv 79060, Ukraine

Abstract

Being motivated by John Tantalo's Planarity Game, we consider straight line plane drawings of a planar graph G with edge crossings and wonder how obfuscated such drawings can be. We define $obf(G)$, the *obfuscation complexity* of G, to be the maximum number of edge crossings in a drawing of G. Relating $obf(G)$ to the distribution of vertex degrees in G, we show an efficient way of constructing a drawing of G with at least $obf(G)/3$ edge crossings. We prove bounds $(\delta(G)^2/24 - o(1))\, n^2 \le obf(G) < 3\, n^2$ for an n-vertex planar graph G with minimum vertex degree $\delta(G) \ge 2$.

The *shift complexity* of G, denoted by $shift(G)$, is the minimum number of vertex shifts sufficient to eliminate all edge crossings in an arbitrarily obfuscated drawing of G (after shifting a vertex, all incident edges are supposed to be redrawn correspondingly). If $\delta(G) \ge 3$, then $shift(G)$ is linear in the number of vertices due to the known fact that the matching number of G is linear. However, in the case $\delta(G) \ge 2$ we notice that $shift(G)$ can be linear even if the matching number is bounded. As for computational complexity, we show that, given a drawing D of a planar graph, it is NP-hard to find an optimum sequence of shifts making D crossing-free.

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1 Introduction

This note is inspired by John Tantalo's Planarity Game [10] (another implementation is available at [13]). An instance of the game is a straight line drawing of a planar graph with many edge crossings. In a move the player is able to shift one vertex of the graph to a new position; the incident edges will be redrawn correspondingly. The objective is to achieve a crossing-free drawing in a possibly smaller number of moves.

Let us fix some relevant terminology. By a *drawing* we will always mean a straight line plane drawing of a graph where no vertex is an inner point of any edge. An *edge crossing* in a drawing D is a pair of edges having a common inner point. The number of edge crossings in D will be denoted by $obf(D)$. We define the *obfuscation complexity* of a graph G to be the maximum $obf(D)$ over all drawings D of G. This graph parameter will be denoted by $obf(G)$.

Given a drawing D of a planar graph G, let $shift(D)$ denote the minimum number of vertex shifts making D crossing-free. The *shift complexity* of G, denoted by $shift(G)$, is the maximum $shift(D)$ over all drawings of G.

Our aim is a combinatorial and a complexity-theoretic analysis of the Planarity Game from the standpoint of a game designer. The latter should definitely have a library of planar graphs G with large $shift(G)$. Generation of planar graphs with large $obf(G)$ is also of interest. Though large obfuscation complexity does not imply large shift complexity (see discussion in Section 4.4), the designer can at least expect that a large $obf(D)$ will be a psychological obstacle for a player to play optimally on D.

A result of direct relevance to the topic is obtained by Pach and Tardos [8]. Somewhat surprisingly, they prove that even cycles have large shift complexity, namely, $n - O((n \log n)^{2/3}) \le shift(C_n) \le n - \lfloor \sqrt{n} \rfloor$.

We first address the obfuscation complexity. In Section 2 we relate this parameter of a graph to the distribution of its vertex degrees. This gives us an efficient way of constructing a drawing D of a given graph G so that $obf(D) \ge obf(G)/3$. As another consequence, we prove that $obf(G) \ge (\delta(G)^2/24 - o(1))n^2$ for an n-vertex planar graph with minimum vertex degree $\delta(G) \ge 2$. On the other hand, we prove an upper bound $obf(G) < 3\,n^2$. In Section 3 we discuss the relationship between the shift complexity of a planar graph and its matching number. We also show that the shift complexity of a drawing is NP-hard to compute. Section 4 contains concluding remarks and questions.

Related work. Investigation of the parameter *shift*(G) is well motivated from a graph drawing perspective. Several results were obtained in this area independently of our work and appeared in [3, 9, 2] soon after the present note was submitted to the journal. The Planarity Game is also mentioned in [3, 9] as a source of motivation.

Goaos et al. [3] independently prove that computing *shift*(D) for a given drawing D is an NP-hard problem, the same result as stated in our Theorem 8. They use a different reduction, allowing them to show that *shift*(D) is even hard to approximate. Our reduction has another advantage: It shows that it is NP-hard to untangle even drawings of as simple graphs as matchings.

Spillner and Wolff [9] and Bose et al. [2] obtain general upper bounds for *shift*(G), which quantitatively improve the classical Wagner-Fáry-Stein theorem (cf. Theorem 4 in Section 3). The stronger of their bounds [2] claims that *shift*$(G) \le n - \sqrt[4]{n/9}$ for any planar G. Even better bounds are established for trees [3] and outerplanar graphs [9]. The series of papers [3, 9, 2] gives also lower bounds on the variant of *shift*(G) for a broader notion of a "bad drawing".

Notation. We reserve n and m for, respectively, the number of vertices and the number of edges in a graph under consideration. We use the standard notation K_n, $K_{s,t}$, and C_n for, respectively, complete graphs, complete bipartite graphs, and cycles. The vertex set of a graph G will be denoted by $V(G)$. By kG we mean the disjoint union of k copies of G. The number of edges emanating from a vertex v is called the *degree* of v and denoted by $\deg v$. The *minimum degree* of a graph G is defined by $\delta(G) = \min_{v \in V(G)} \deg v$. A set of pairwise non-adjacent vertices (resp., edges) is called an *independent set* (resp., a *matching*). The maximum cardinality of an independent set (resp., a matching) in a graph G is denoted by $\alpha(G)$ (resp., $\nu(G)$) and called the *independence number* (resp., the *matching number*) of G. A graph is *k-connected* if it stays connected after removal of any $k - 1$ vertices.

2 Estimation of the obfuscation complexity

Note that $obf(G)$ is well defined for an arbitrary, not necessary planar graph G. As a warm-up, consider a few examples.

$obf(K_n) = \binom{n}{4}$. Indeed, let D be a drawing of K_n. $obf(D)$ is computable as follows. We start with the initial value 0 and, tracing through all

pairs $\{e, e'\}$ of non-adjacent edges, increase it by 1 once e and e' cross. Consider the set S of 4 endpoints of e and e'. In fact, S corresponds to exactly 3 pairs of edges. If the convex hull of S is a triangle, then none of these three pairs is crossing. If it is a quadrangle, then 1 of the three pairs is crossing and 2 are not. It follows that $obf(D)$ does not exceed the number of all possible S. This upper bound is attained if every S has a quadrangular hull, for instance, if the vertices of D lie on a circle.

$obf(K_{s,t}) = \binom{s}{2}\binom{t}{2}$. The upper bound is provable by the same argument as above, where a 4-point set S has 2 points in the s-point part of $V(D)$ and 2 points in the t-point part. Such an S corresponds to 2 pairs of non-adjacent edges, at most 1 of which is crossing. This upper bound is attained if we put the two vertex parts of $K_{s,t}$ on two parallel lines.

$obf(C_n) = n(n-3)/2$ if n is odd. The value of $n(n-3)/2$ is attained by the n-pointed star drawing of C_n. This is the maximum by a simple observation: $n(n-3)/2$ is the total number of pairs of non-adjacent edges in C_n.

Let us state the upper bound argument we just used for the odd cycles in a general form. Given a graph G with m edges, let

$$\epsilon(G) = \binom{m}{2} - \sum_{v \in V(G)} \binom{\deg v}{2}.$$

Note that $\epsilon(G) = \frac{1}{2}(m(m+1) - \sum_v \deg^2 v)$, where the latter term is closely related to the variance of the vertex degrees. Since $\epsilon(G)$ is equal to the number of pairs of non-adjacent edges in G, we have $obf(G) \le \epsilon(G)$. Notice also a lower bound in terms of $\epsilon(G)$.

Theorem 1. $\epsilon(G)/3 \le obf(G) \le \epsilon(G)$. *Moreover, a drawing D of G with $obf(D) \ge \epsilon(G)/3$ is efficiently constructible.*

Proof. Fix an arbitrary n-point set V on a circle. We use the probabilistic method to prove that there is a drawing D with $V(D) = V$ having at least $\epsilon(G)/3$ edge crossings. Let $\mathbf{D}$ be a random straight line embedding of G with $V(\mathbf{D}) = V$, which is determined by a random map of $V(G)$ onto V. For each pair e, e' of non-adjacent vertices of G, we define a random variable $X_{e,e'}$ by

$X_{e,e'} = 1$ if e and e' cross in $\mathbf{D}$ and $X_{e,e'} = 0$ otherwise. Let S be a 4-point subset of V. Under the condition that the set of endpoints of e and e' in $\mathbf{D}$ is S, these edges cross one another in $\mathbf{D}$ with probability 1/3. It follows that $X_{e,e'} = 1$ with probability 1/3. Note that $\mathit{obf}(\mathbf{D}) = \sum_{\{e,e'\}} X_{e,e'}$. By linearity of the expectation, we have $\mathbb{E}\left[\mathit{obf}(\mathbf{D})\right] = \sum_{\{e,e'\}} \mathbb{E}\left[X_{e,e'}\right] = \frac{1}{3}\,\epsilon(G)$ and hence $\mathit{obf}(D) \ge \frac{1}{3}\,\epsilon(G)$ for at least one instance D of $\mathbf{D}$. Such a D is efficiently constructible by standard derandomization techniques, namely, by the method of conditional expectations, see, e.g., [1, Chapter 15]. ∎

As a consequence of Theorem 1, we have $\mathit{obf}(G) = \Theta(n^2)$ for a planar G whenever $\delta(G) \ge 2$ (the latter condition excludes the cases like $\mathit{obf}(K_{1,s}) = 0$). Indeed, $\epsilon(G) < \frac{9}{2}\,n^2$ because $m < 3n$ for any planar graph. This bound is sharp in the sense that $\epsilon(G) \ge \frac{9}{2}\,n^2 - O(n)$ for maximal planar graphs of bounded vertex degree. A sharp lower bound for $\epsilon(G)$ is stated below.

Theorem 2. *$\epsilon(G) \ge \left(\frac{\delta(G)^2}{8} - o(1)\right) n^2$ for a planar graph G with $\delta(G) \ge 2$. The constant $\delta(G)^2/8$ cannot be better here.*

Proof. Let $A_k(G) = \{\, v \in V(G) : \deg v < k \}$ and denote

$$a_k(G) = |A_k(G)| \quad \text{and} \quad s_k(G) = \sum_{v \in V(G) \setminus A_k(G)} \deg v.$$

West and Will [12] prove that, if $k \ge 12$, then for every planar G on $n \ge \frac{3}{2}k - 1$ vertices we have

$$a_k(G) \ge \frac{(k-8)n + 16}{k-6}$$

and

$$s_k(G) < 2\,n - 16 + \frac{12(n-8)}{k-6}.$$

We begin with the bound

$$\epsilon(G) > \frac{1}{2}\left(m^2 - \sum_{v \in V(G)} \deg^2 v\right).$$

Set $\delta = \delta(G)$. Let $\sigma = s_k(G)/n$ (to simplify the notation, we do not indicate the dependence of σ on k). Suppose that k is large enough, namely, $k \ge 14$.

Note that $0 \leq \sigma < 2 + 12/(k-6)$. We now estimate m from below and $\sum_v \deg^2 v$ from above.

$$m = \frac{1}{2}\sum_v \deg v = \frac{1}{2}\left(\sum_{v \in A_k(G)} \deg v + \sum_{v \notin A_k(G)} \deg v\right)$$
$$\geq \frac{1}{2}\left(\delta(G)a_k(G) + s_k(G)\right) > \frac{1}{2}\left(\frac{\delta(k-8)}{k-6} + \sigma\right) n.$$

Furthermore,

$$\sum_v \deg^2 v = \sum_{v \in A_k(G)} \deg^2 v + \sum_{v \notin A_k(G)} \deg^2 v < (k-1)^2 n + f(\sigma) n^2,$$

where

$$f(\sigma) = \begin{cases} 2 + (\sigma-2)^2 & \text{if } \ 2 \leq \sigma < 2 + 12/(k-6), \\ 1 + (\sigma-1)^2 & \text{if } \ 1 \leq \sigma < 2, \\ \sigma^2 & \text{if } \ 0 \leq \sigma < 1. \end{cases}$$

Thus,

$$\epsilon(G) > g(\sigma)\, n^2 - \frac{(k-1)^2}{2} n, \ \text{ where } \ g(\sigma) = \frac{1}{2}\left(\frac{1}{4}\left(\frac{\delta(k-8)}{k-6} + \sigma\right)^2 - f(\sigma)\right).$$

A routine calculation shows that

$$\min\left\{ g(\sigma) : 0 \leq \sigma < 2 + \frac{12}{k-6}\right\} = g(0) = \frac{\delta^2}{8}\left(\frac{k-8}{k-6}\right)^2.$$

We conclude that

$$\epsilon(G) > \frac{\delta^2}{8}\left(\frac{k-8}{k-6}\right)^2 n^2 - \frac{(k-1)^2}{2} n > \left(\frac{\delta^2}{8} - \frac{\delta^2}{2(k-6)} - \frac{(k-1)^2}{2n}\right) n^2$$

whenever $k \geq 14$ and $n \geq \frac{3}{2}k - 1$. Recall that $\delta(G) \leq 5$ for any planar G. If we make k a function of n that grows to the infinity slower than $\sqrt{n}$, then the factor in front of n^2 becomes $\delta^2/8 - o(1)$ and we arrive at the claimed bound.

The optimality of the constant $\delta^2/8$ is ensured by regular planar graphs (i.e., cycles and cubic, quartic, and quintic planar graphs). ∎

As was already mentioned, for planar graphs we have $\mathit{obf}(G) \le \epsilon(G) < \frac{9}{2}\,n^2$, where the bound for $\epsilon(G)$ cannot be improved. However, for $\mathit{obf}(G)$ we can do somewhat better.

Theorem 3. *$\mathit{obf}(G) < 3\,n^2$ for a planar graph G on n vertices.*

Proof. Note that, if K is a subgraph of H, then $\mathit{obf}(K) \le \mathit{obf}(H)$. It therefore suffices to prove the theorem for the case that G is a maximal planar graph, that is, a triangulation. Let E be a (crossing-free, not necessary straight line) plane embedding of G. Denote the number of triangular faces in E by t and note that $3t = 2m$. Based only on facial triangles, let us estimate from below the number of non-crossing edge pairs in an arbitrary straight line drawing D of G. Let P denote the set of all pairs of adjacent edges occurring in facial triangles. Here we have $|P| = 3t$ edge pairs which are non-crossing in D. Furthermore, for each pair of edge-disjoint facial triangles $\{T, T'\}$ we take into account pairs of non-crossing edges $\{e, e'\}$ with e from T and e' from T'. Since at most $3t/2$ pairs of facial triangles can share an edge, there are at least $\binom{t}{2} - \frac{3t}{2}$ such $\{T, T'\}$. We split this amount into two parts. Let A consist of vertex-disjoint $\{T, T'\}$ and B consist of $\{T, T'\}$ sharing one vertex. As easily seen, every $\{T, T'\}$ in A gives us at least 3 edge pairs $\{e, e'\}$ which are non-crossing in D. Every $\{T, T'\}$ in B contributes at least 2 pairs of non-adjacent edges and exactly 4 pairs of adjacent edges. However, 2 of the latter 4 edge pairs can participate in P. We conclude that in D there are at least $|P| + (3|A| + 4|B|)/4$ non-crossing edge pairs. The factor of $1/4$ in the latter term is needed because an edge pair $\{e, e'\}$ can be contributed by 4 triangle pairs $\{T, T'\}$. Thus,

$$\mathit{obf}(D) \le \binom{m}{2} - 3t - \frac{3}{4}\left(\binom{t}{2} - \frac{3t}{2}\right) < \frac{1}{2}\,m^2 - \frac{3}{8}\,t^2 = \frac{1}{3}m^2.$$

Since $m < 3n$ as a simple consequence of Euler's formula, we have $\mathit{obf}(D) < 3n^2$. As D is arbitrary, the bound for $\mathit{obf}(G)$ follows. ∎

3 Estimation of the shift complexity

A basic fact about $\mathit{shift}(G)$ is that this number is well defined.

Theorem 4 (Wagner, Fáry, Stein (see, e.g., [6])). *Every planar graph G has a straight line plane drawing. In other words, shift$(G) \le n-3$ if $n \ge 3$.*

If we seek for lower bounds, the following example is instructive despite its simplicity: $\mathit{shift}(mK_2) = m-1$. It immediately follows that

$$\mathit{shift}(G) \ge \nu(G) - 1.$$

Theorem 5. *Let G be a connected planar graph on n vertices.*

1. *If $\delta(G) \ge 3$ (in particular, if G is 3-connected) and $n \ge 10$, then $\mathit{shift}(G) \ge (n-1)/3$.*
2. *If G is 4-connected, then $\mathit{shift}(G) \ge (n-3)/2$.*
3. *There is an infinite family of connected planar graphs G with $\delta(G) = 2$ and $\mathit{shift}(G) \le 2$.*

Proof. Item 1 follows from the fact that, under the stated conditions on G, we have $\nu(G) \ge (n+2)/3$ (Nishizeki-Baybars [5]). Item 2 is true because every 4-connected planar G is Hamiltonian (Tutte [11]) and hence $\nu(G) \ge (n-1)/2$ in this case. Item 3 is due to the bound $\mathit{shift}(K_{2,s}) \le 2$. The latter follows from the elementary fact of plane geometry stated in Lemma 6 below. ∎

Lemma 6. *For any finite set of points Z there are two points x and y such that the segments with one endpoint in $\{x, y\}$ and the other in Z do not cross each other and have no inner points in Z.*

Proof. Let L denote the set of all lines going through at least two points in Z. Fix the direction "upward" not in parallel to any line in L. Pick up x above every line in L and y below every line in L. ∎

The next question we address is this: How close is relationship between $\mathit{shift}(G)$ and $\nu(G)$? By Theorem 5, if $\delta(G) \ge 3$ then both graph parameters are linear. However, if $\delta(G) \le 2$, the existence of a large matching is not the only cause of large shift complexity.

Theorem 7. *There is a planar graph G_s on $3s+3$ vertices with $\delta(G_s) = 2$ such that $\nu(G_s) = 3$ and $\mathit{shift}(G_s) \ge 2s-6$.*

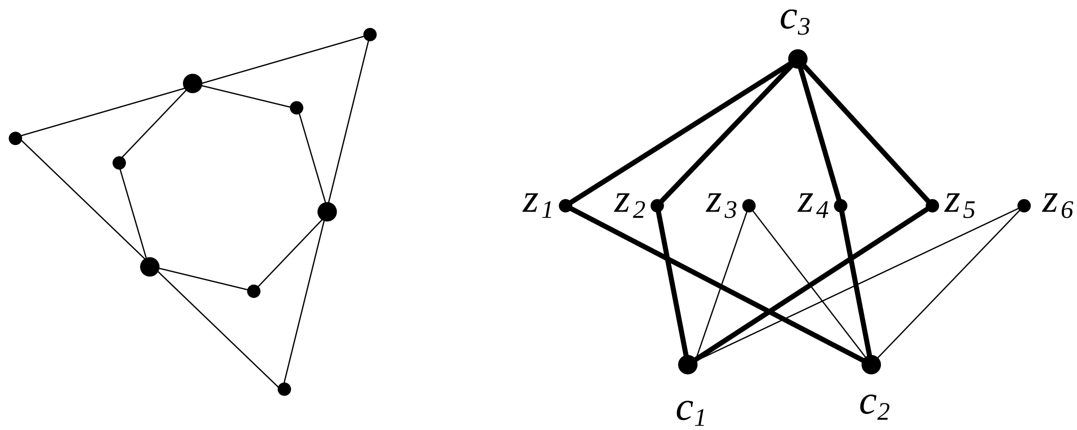

Figure 1: G_2 and F in D_2.

Proof. A suitable G_s can be obtained as follows: take the multigraph which is triangle with multiplicity of every edge s and make it graph by inserting a new vertex in each of the $3s$ edges (see Fig. 1). Using Lemma 6, it is not hard to show that $\mathit{shift}(G_s) \leq 2s + 3$. We now construct a drawing D_s of G_s with $\mathit{shift}(D_s) \geq 2s - 6$. Put vertices $z_1, \ldots, z_{3s}$ in this order in a line and the remaining vertices c_0, c_1, c_2 somewhere else in the plane. Connect z_i with c_j iff $j \neq i \bmod 3$. Therewith D_s is specified. Denote the fragment of D_s induced on $\{z_1, z_2, z_4, z_5, c_0, c_1, c_2\}$ by F. It is not hard to see that F cannot be disentangled by moving only c_0, c_1, and c_2. In fact, if in place of z_1, z_2, z_4, z_5 we take any quadruple z_i, z_j, z_k, z_l with $i < j < k < l$, $i \equiv k \pmod 3$, and $j \equiv l \pmod 3$, this will give us a fragment completely similar to F. To destroy all such fragments, we need to move at least two vertices in every triple $z_{3h+1}, z_{3h+2}, z_{3h+3}$ $(0 \leq h < s)$ with possible exception for at most 3 of them. Therefore, making $2(s - 3)$ shifts is unavoidable. ∎

Finally, we prove a complexity result.

Theorem 8. *Computing the shift complexity of a given drawing is an NP-hard problem.*

Proof. In fact, this hardness result is true even for drawings of graphs mK_2. Given such a drawing D, consider its intersection graph S_D whose vertices are the edges of D with e and e' adjacent in S_D iff they cross one another in D. Since computing the independence number of intersection graphs of segments in the plane is known to be NP-hard (Kratochvíl-Nešetřil [4]), it

suffices for us to express $\alpha(S_D)$ as a simple function of *shift*(D). Fix an optimal way of untangling D and denote the set of edges whose position was not changed by E. Clearly, E is an independent set in S_D and hence *shift*$(D) \geq m - |E| \geq m - \alpha(S_D)$. On the other hand, *shift*$(D) \leq m - \alpha(S_D)$. Indeed, fix an independent set I in S_D of the maximum size $\alpha(S_D)$. Then D can be untangled this way: we leave the edges in I unchanged and shrink each edge not in I by shifting one endpoint sufficiently close to the other endpoint. Thus, $\alpha(S_D) = m -$ *shift*(D), as desired. ∎

4 Concluding remarks and problems

1. By Theorem 1 we have $\frac{1}{3}\,\epsilon(G) \leq obf(G) \leq \epsilon(G)$. The upper bound cannot be improved in general as $obf(C_n) = \epsilon(C_n)$ for odd n. Can one improve the factor of $\frac{1}{3}$ in the lower bound?

2. By Theorems 1, 2, and 3 we have $(\delta(G)^2/24 - o(1))n^2 \leq obf(G) \leq 3\,n^2$ where $\delta(G) \geq 2$ is necessary for the lower bound. Optimize the factors in the left and the right hand sides.

3. As follows from the proof of Theorem 1, there is an n-point set V (in fact, this can be an arbitrary set on the border of a convex body) with the following property: Every graph G of order n has a drawing D with $V(D) = V$ such that $obf(D) \geq \frac{1}{3} obf(G)$. Can this uniformity result be strengthened? Is there an n-point set V on which one can attain $obf(D) = obf(G)$ for all n-vertex G?

4. The following remarks show that the obfuscation and the shift complexity of a drawing have, in general, rather independent behavior.

Maximum $obf(D)$ does not imply maximum shift(D). Consider $3K_{1,s}$, the union of 3 disjoint copies of the s-star. It is not hard to imagine how a drawing attaining $obf(3K_{1,s}) = 3s^2$ should look (where every two non-adjacent edges cross) and it becomes clear that such a drawing can be untangled just by 2 shifts. However, *shift*$(3K_{1,s}) \geq s$ is provable similarly to Theorem 7 (an upper bound *shift*$(3K_{1,s}) \leq s + 2$ follows from Lemma 6).

Maximum shift(D) does not imply maximum $obf(D)$. The simplest example is given by a drawing of the disjoint union of K_2 and $K_{1,2}$ with only one edge crossing.

Large obf(D) does not imply large shift(D). This can be shown by drawings of *obf*$(K_{2,s})$. Indeed, we know that *obf*$(K_{2,s}) = \binom{s}{2}$ from Section 2 and *shift*$(K_{2,s}) \le 2$ from Section 3 (the latter bound is exact if $s \ge 4$).

Large shift(D) does not imply large obf(D). Pach and Tardos [8, Fig. 2] show a drawing D of the cycle C_n with linear *shift*(D) and *obf*$(D) = 1$.

5. In spite of the observation we just made that large *obf*(D) does not imply large *shift*(D), in some interesting cases it does. Pach and Solymosi [7] prove that every system S of m segments in the plane with $\Omega(m^2)$ crossings has two disjoint subsystems S_1 and S_2 with both $|S_1| = \Omega(m)$ and $|S_2| = \Omega(m)$ such that every segment in S_1 crosses all segments in S_2. As *shift*$(S) \ge \min\{|S_1|, |S_2|\}$, this result has an interesting consequence: If D is a drawing of mK_2 with *obf*$(D) = \Omega(m^2)$, then *shift*$(D) = \Omega(m)$.

6. Theorem 8 shows that computing *shift*(D) for a drawing D of a graph G can be hard even in the cases when computing *shift*(G) is easy. Is *shift*(G) hard to compute in general? Theorem 1 shows that *obf*(G) is polynomial-time approximable within a factor of 3. Is exact computation of *obf*(G) NP-hard (Amin Coja-Oghlan)?

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